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ALMOST EVERYWHERE SUMMABILITY ON NILMANIFOLDS[1]

BY

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ABSTRACT. Let G be a stratified, nilpotent Lie group and let L be a homogeneous sublaplacian on G. Let $E(\lambda)$ denote the spectral resolution of L on $L^2(G)$. Given a function K on $\mathbf{R}^+$, define the operator T_K on $L^2(G)$ by $T_k f = \int_0^\infty K(\lambda)\, dE(\lambda) f$. Sufficient conditions on K to imply that T_K is bounded on $L^1(G)$ and the maximal operator $K^*\varphi(x) = \sup_{t>0} | T_{K_t}\varphi(x) |$ (where $K_t(\lambda) = K(t\lambda)$) is of weak type $(1, 1)$ are given. Picking a basis $e_0, e_1, \ldots$ of $L^2(G/\Gamma)$ (Γ being a discrete cocompact subgroup of G) consisting of eigenfunctions of L, we obtain almost everywhere and norm convergence of various summability methods of $\Sigma(\varphi, e_j)e_j$, $\varphi \in L^p(G/\Gamma)$, $1 \leqslant p < \infty$.

Let G be a nilpotent Lie group and Γ a discrete, cocompact subgroup of G. There is a unique G-invariant probability measure $d\dot{x}$ on G/Γ. $L^2(G/\Gamma)$ can be written as a direct sum of irreducible (with finite multiplicities) G-invariant subspaces

$$L^2(G/\Gamma) = \bigoplus \sum_j \mathcal{H}_j. \tag{0.1}$$

In other words, if P_j denotes the orthogonal projection of $L^2(G/\Gamma)$ onto $\mathcal{H}_j$ we have

$$f = \sum_j P_j f \qquad (f \in L^2(G/\Gamma)) \tag{0.2}$$

where the series is convergent in $L^2(G/\Gamma)$. The question of whether there are other summability methods of (0.2) for $f \in L^p(G/\Gamma)$, or whether the series is convergent, e.g., uniformly for functions with some degree of smoothness, has long been considered and is the starting point of this paper. (Some facts concerning these questions can be found in **[10]**.) On the other hand, G/Γ is a compact manifold, so if Δ is an elliptic selfadjoint operator on G/Γ, $L^2(G/\Gamma)$ has a basis of eigenfunctions for Δ. If Δ is G-invariant, the elements of this basis can be selected from the subspaces in (0, 1). The eigenfunction expansions for such operators have been extensively studied, and the asymptotic of the eigenvalues and various summability methods of the eigenfunction expansions are known (cf., e.g., **[14]**).

In this paper we consider the homogeneous sublaplacian L (which is not elliptic) on a stratified nilpotent Lie group G. If $E(\lambda)$ is the spectral resolution of L on $L^2(G)$ and K is a function on $\mathbf{R}^+$, set

$$T_K f = \int_0^\infty K(\lambda)\, dE(\lambda) f \qquad (f \in L^2(G)).$$

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Following the program of E. M. Stein [12], we consider conditions on K such that T_K is bounded on $L^1(G)$ and such that the maximal operator

$$K^*\varphi(x) = \sup_{t>0} | T_{K_t}\varphi(x) |$$

(where $K_t(\lambda) = K(t\lambda)$) is of weak type (1, 1). This is done by functional calculus on related commutative Banach algebras as in [5]. This leads to the almost everywhere convergence to f, for $f \in L^p(G)$, $1 \leqslant p < \infty$, of, e.g., the Bochner-Reisz means

$$\int_0^R (1 - \lambda/R)^N \, dE(\lambda)\, f \tag{0.3}$$

for N sufficiently large. Giancarlo Mauceri [7] has studied this problem on the Heisenberg group by different methods.

Picking a basis $e_0, e_1, \dots$ of $L^2(G/\Gamma)$ consisting of eigenfunctions of L, the corresponding theorems give almost everywhere and norm convergence of various summability methods of $\Sigma_j(\varphi, e_j)e_j$ ($\varphi \in L^p(G/\Gamma)$). Considering φ's in one summand of (0.1) and carrying it over to $\mathbf{R}^k$, we obtain almost everywhere convergence of Abel, Cesàro, etc. summability methods for multiple Hermite expansions. For example, for $f \in L^p(R)$, $1 \leqslant p < \infty$, and $\alpha > 9$,

$$f(x) = \lim_{N \to \infty} \sum_{j=0}^{N} (1 - j/N)^\alpha (f, \varphi_j)\varphi_j(x) \quad \text{(a.e)}$$

where $\varphi_0, \varphi_1, \dots$ are Hermite functions. Related results can be found in [8 and 9].

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1. Let G be a stratified nilpotent Lie group, i.e. if $\mathfrak{g}$ denotes the Lie algebra of G we have

$$\mathfrak{g} = \bigoplus \sum_j \mathfrak{g}_j,$$

where $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. For every $s > 0$ the mapping $X_j \to s^j X_j$ for $X_j \in \mathfrak{g}_j$ defines an automorphism of $\mathfrak{g}$. We denote by δ_s the corresponding automorphism of G, and by $| \cdot |$ a homogeneous gauge on G. For a fixed compact, symmetric neighborhood U of e in G we define

$$\tau(x) = \min\{n \mid x \in U^n\}.$$

One then has (cf. [3, 4, 6])

$$|xy| \leqslant c(|x| + |y|) \tag{1.1}$$

for some c and all $x, y \in G$, and

$$c|x| \leqslant \tau(x) \leqslant a|x| + b \tag{1.2}$$

for some $a, b, c > 0$ and all $x \in G$. The function τ is subadditive ($\tau(xy) \leqslant \tau(x) + \tau(y)$), and, hence, $\omega = 1 + \tau$ is a submultiplicative, symmetric weight on G. The space

$$L^1_{\omega^l} = \left\{ f \in L^1(G) \;\middle|\; \int |f(x)|\, \omega^l(x)\, dx < \infty \right\}$$

is clearly a Banach*-algebra (under convolution).

For $s > 0$, we set $B_s = \{x \in B \mid |x| \leq s\}$. One has then that $|B_s| = s^r |B_1|$, where $|E|$ denotes Haar measure of E, and r, the homogeneous dimension of G, is given by $r = \Sigma_j j \dim \mathfrak{g}_j$.

For a function f on G we write

$$\rho_s f(x) = s^{-r/2} f\big(\delta_{s^{-1/2}}(x)\big)$$

and observe that ρ_s is a norm preserving automorphism of the Banach*-algebra $L^1(G)$. We also observe:

(1.3) If $f \in L^1(G)$ and $\int f = 1$, then $\{\rho_s f\}_{s \to 0}$ is an approximate identity.

Let $X_1, \dots, X_d$ be a basis for $\mathfrak{g}$ such that $X_1, \dots, X_k$ is a basis for $\mathfrak{g}_1$. Consider the operators

$$-\Delta = X_1^2 + \cdots + X_d^2, \qquad -L = X_1^2 + \cdots + X_k^2$$

(Δ is, of course, invariant under any orthogonal change of basis in $\mathfrak{g}$). Let

$$\|\phi\|_s = \|(I + \Delta)^{s/2} \phi\|_{L^2}.$$

The classical Sobolev inequality states that

$$\|\phi\|_{L^\infty} \leq C \|\phi\|_{[d/2]+1}.$$

For each s, there is a $\sigma(s)$, depending only on $\mathfrak{g}$, such that

$$\|\phi\|_s \leq C \|(I + L)^{\sigma(s)} \phi\|_{L^2}, \tag{1.4}$$

and so, there is a smallest integer S such that

$$\|\phi\|_{L^\infty} \leq C \|(I + L)^S \phi\|_{L^2}. \tag{1.4$'$}$$

The numbers S and $\sigma := \sigma(1)$ will play an important role in the following considerations.

The distributions $\phi \to -\Delta\phi(e)$ and $\phi \to -L\phi(e)$ are dissipative, and thus define, by convolution, infinitesimal generators of unique semigroups of measures on G, denoted $\{P_t\}_{t>0}$ and $\{p_t\}_{t>0}$, respectively. Furthermore, P_t and p_t are selfadjoint, nonnegative functions in $L^1(G)$ (cf. e.g. [5]). Since, for $s > 0$, ρ_s is an automorphism of $L^1(G)$, $\{\rho_s p_t\}_{t>0}$ is a convolution semigroup. One easily checks that its infinitesimal generator is sL. Thus

$$\rho_s p_t = p_{st}. \tag{1.5}$$

Let $\underline{\mathfrak{A}}$ be the Banach*-algebra generated by $\{p_t\}_{t>0}$ in $L^1(G)$, i.e.

$$\underline{\mathfrak{A}} = \operatorname{span}\{p_t \mid t > 0\}^{-\|\cdot\|_{L^1}}.$$

We write $\underline{\mathfrak{A}}_l = \underline{\mathfrak{A}} \cap L^1_{\omega^l}$. The Gelfand space $\underline{\mathfrak{A}}$ or $\underline{\mathfrak{A}}_l$ is homeomorphic to $\mathbf{R}^+$. Letting $\hat{f}$ denote the Gelfand transform for f in $\underline{\mathfrak{A}}$ or $\underline{\mathfrak{A}}_l$, one has (cf. [5])

$$\hat{p}_t(\lambda) = e^{-t\lambda}. \tag{1.6}$$

By virtue of (1.5), ρ_s maps $\underline{\mathfrak{A}}$ into itself, and (1.5) and (1.6) together imply

$$(\rho_s \hat{f})(\lambda) = \hat{f}(s\lambda) \tag{1.7}$$

for $f \in \mathcal{Q}$. Since I: $f \to \int f(x)\,dx$ is a multiplicative functional on $\mathcal{Q}$, and $I(\rho_s f) = I(f)$, (1.7) implies that $I(f) = \hat{f}(0)$. Also, if $f \in \mathcal{Q} \cap L^2(G)$, then

$$\| f \|_{L^2}^2 = c \int | \hat{f}(\lambda) |^2 \lambda^{(r-2)/2}\, d\lambda. \tag{1.8}$$

To see this, note that by (1.5),

$$\| p_t \|_{L^2}^2 = t^{-r/2} \| p_1 \|_{L^2}^2.$$

Hence,

$$(p_s, p_t) = \| p_{(s+t)/2} \|_{L^2}^2 = ((s+t)/2)^{-r/2} \| p_1 \|_{L^2}^2. \tag{1.9}$$

But one also has

$$\begin{aligned} \int \hat{p}_s(\lambda)\hat{p}_t(\lambda)\lambda^{(r-2)/2}\, d\lambda &= \int e^{-(s+t)\lambda}\lambda^{(r-2)/2}\, d\lambda \\ &= (s+t)^{-r/2} \int e^{-\lambda}\lambda^{(r-2)/2}\, d\lambda, \end{aligned} \tag{1.10}$$

which shows that

$$(p_s, p_t) = c \int \hat{p}_s(\lambda)\hat{p}_t(\lambda)\lambda^{(r-2)/2}\, d\lambda.$$

Let B denote a Banach*-algebra. For $f = f^* \in B$ we let B_f be the commutative Banach*-algebra generated by f. For $g \in B_f$, we denote by $\mathrm{Sp}_{B_f}(g)$ its spectrum in B_f. A function F: $\mathbf{R} \to \mathbf{C}$ is said to operate on f if $\mathrm{Sp}_{B_f}(f) \subseteq \mathbf{R}$ and there is a $g \in B_f$ such that $F\hat{f} = \hat{g}$. The classical criterium for a function to operate on $f \in B$ (cf. e.g. [**2**]) is as follows. For $f \in B$ set

$$e(f) = \sum_{k=1}^{\infty} \frac{1}{k!} f^k.$$

If

$$\| e\left(\sqrt{-1}\, nf\right) \|_B = O(n^a) \qquad (n \to \infty), \tag{1.11}$$

then $\mathrm{Sp}_{B_f}(f) \subseteq \mathbf{R}$ and, for $F \in C^l(\mathbf{R})$, $l > a + 1$, with $F(0) = 0$, F operates on f.

It is well known that both Δ and L are essentially selfadjoint operators on $L^2(G)$. Let $E(\lambda)$ be the spectral resolution of L, i.e.

$$Lf = \int_0^\infty \lambda\, dE(\lambda) f \qquad (f \in D(L)).$$

Then

$$p_t^* f = \int_0^\infty e^{-\lambda t}\, dE(\lambda) f.$$

Let K be a bounded function on $\mathbf{R}^+$ and set

$$T_K \phi = \int_0^\infty K(\lambda)\, dE(\lambda)\, \phi \qquad (\phi \in L^2(G)).$$

Then T_K is a bounded convolution operator, and, by (1.4′), the mapping $\phi \to T_K\phi(e)$ is a distribution. Thus there is a distribution k such that $T_K\phi = k * \phi$ for $\phi \in L^2(G)$. If $k \in \mathcal{Q}$ then $k = \hat{K}$. Our first theorem gives sufficient conditions on K to imply that $k \in L^1 \cap L^2$.

THEOREM 1.12. *For fixed* $l \geqslant 0$ *we consider the condition* $(K \cdot a)$, $K \in C^N(\mathbf{R}^+)$, *where* $N > r/2 + l + 1$ *and* $\sup_{\lambda \geqslant 0} | K^{(j)}(\lambda)(1+\lambda)^{(a+N)S+1} | < \infty$ *for* $j = 0, 1, \ldots, N$. *Then*:

(i) *if* $a = 0$, *there is a* $k \in \underline{\mathcal{Q}}_l$ *with* $\hat{k} = K$; *and*

(ii) *if* $a = 4$, *there is a* $k \in \underline{\mathcal{Q}}$ *with* $\hat{k} = K$ *and* $\sup_x | k(x)\omega^l(x) | < \infty$.

The proof of Theorem 1.12 requires the following lemmas.

LEMMA 1.13. *There is an* $\varepsilon > 0$ *such that* $\int p_1(x) e^{\varepsilon|x|^2}\, dx < \infty$.

PROOF. It is known (cf. e.g. [4]) that

$$\int p_1(x) e^{s\tau(x)}\, dx \leqslant c e^{as^2} \qquad (s > 0).$$

The proof is completed by either "completing the square" or employing the trick used in [13, p. 277].

LEMMA 1.14. *Let* $R_1 = \int_0^\infty e^{-t} p_t\, dt$. *There is an* $\alpha > 0$ *such that*

$$\left(R_1, e^{\alpha|x|}\right) = \int R_1(x) e^{\alpha|x|}\, dx < \infty.$$

PROOF.

$$\begin{aligned}\left(R_1, e^{\alpha|x|}\right) &= \int_G \int_0^\infty e^{-t} e^{\alpha|x|} t^{-r/2} p_1\left(\delta_{t-1/2} x\right) dt\, dx \\ &= \int_G \int_0^\infty e^{-t+\alpha t^{1/2}|x|}\, dt\, p_1(x)\, dx \leqslant c \int_G e^{\alpha'|x|^2} p_1(x)\, dx\end{aligned}$$

for $\alpha' > 2\alpha$, $C = c(\alpha)$.

LEMMA 1.15. $R_1^* = R_1$ *and* $R_1^S \in L^2(G)$.

PROOF.

$$| (R_1^S, \check{\phi}) | = | R_1^S * \phi(e) | \leqslant C \| (1+L)^S (R_1^S * \phi) \|_{L^2} = C \| \phi \|_{L^2}.$$

LEMMA 1.16. *Let* $f = f^* \in \underline{\mathcal{Q}} \cap L^2(G)$ *with* $(|f|, e^{\alpha|x|}) < \infty$ *for some* $\alpha > 0$. *Let* $a, b \in \mathbf{R}$ *such that* $a < \hat{f}(\lambda) < b$ *for all* $\lambda \in \mathbf{R}^+$. *Let* $F \in C^N(a, b)$ *with* $F(0) = 0$. *If* $N > l + r/2 + 1$, *then* $F \circ f \in \underline{\mathcal{Q}}_l$ *and*

$$\| F \circ f \|_{\underline{\mathcal{Q}}_l} \leqslant C_f \| F \|_{C^N(a,b)}.$$

PROOF. Let $\phi(x) = e^{\beta a \tau(x)}$ where β and a are such that $a\tau(x) \geqslant |x|$ for $x \neq e$ and $\beta < \alpha a$. For each positive integer n we have

$$\begin{aligned}\| e\left(\sqrt{-1}\, nf\right) \|_{\underline{\mathcal{Q}}_l} = &\int_{U^m} | e\left(\sqrt{-1}\, nf\right) | (x) \omega^l(x)\, dx \\ &+ \int_{G/U^m} | e\left(\sqrt{-1}\, nf\right) | (x) e^{\beta\tau(x)} \omega^l(x) e^{\beta\tau}(x).\end{aligned}$$

Since $\omega^l(x) \leqslant (1+m)^l$ for $x \in U^m$, and since $\|e(\sqrt{-1}\,nf)\|_{L^1_\phi} \leqslant e^{\|f\|}L^1_\phi$ for any submultiplicative function ϕ, we have

$$\left\|e\left(\sqrt{-1}\,nf\right)\right\|_{\underline{\mathscr{Q}}_l} \leqslant (1+m)^l\,|U^m|^{1/2}\left\|e\left(\sqrt{-1}\,nf\right)\right\|_{L^2} + \exp\left(n\|f\|_{L^1_\psi}\right),$$

where $\psi(x) = e^{\beta\tau(x)}\omega^l(x)$. For $m = ([c/\beta]+1)n$,

$$\left\|e\left(\sqrt{-1}\,nf\right)\right\|_{\underline{\mathscr{Q}}_l} \leqslant c'\left(1+([c/\beta]+1)n\right)^{l+r/2} n\|f\|_{L^2}.$$

PROOF (THEOREM 1.12). Suppose K satisfies $(K\cdot 0)$ and let

$$F(\lambda) = \begin{cases} K(\lambda^{-1/S}-1), & \lambda > 0, \\ 0, & \lambda \leqslant 0, \end{cases}$$

$F(0) = 0$ and $F \in C^N(\mathbf{R})$. By Lemmas 1.14 and 1.15, R_1^S satisfies the conditions on f in Lemma 1.16. Thus, since $(R_1^S)\hat{\ }(\lambda) = (1+\lambda)^{-S}$,

$$K = F \circ \left(R_1^S\right)\hat{} = \hat{k} \quad \text{for some } k \in \underline{\mathscr{Q}}_l.$$

If K satisfies $(K\cdot 4)$ and $K_0(\lambda) = (1+\lambda)^{4S}K(\lambda)$, then K_0 satisfies $(K\cdot 0)$. By (i) there is a $k_0 \in \underline{\mathscr{Q}}_l$ such that $\hat{k}_0(\lambda) = K_0(\lambda)$. Thus, since $R_1^S \in \underline{\mathscr{Q}}_l$, $k = R_1^S * k_0 \in \underline{\mathscr{Q}}_l$ and $\hat{k} = K$. Also,

$$\begin{aligned} |k(x)\omega^l(x)| &\leqslant \int R_1^{4S}(y)\,|k_0(y^{-1}x)|\,\omega^l(x)\,dy \\ &\leqslant \int R_1^{4S}(y)\,|k_0(y^{-1}x)|\,\omega^l(y^{-1}x)\omega^l(x)\,dy, \end{aligned}$$

and

$$\begin{aligned} R_1^{4S}(y)\omega^l(y) &\leqslant \int R_1^{2S}(x)\omega^l(x)R_1^{2S}(x^{-1}y)\omega^l(x^{-1}y)\,dx \\ &\leqslant \int \left(R_1^{2S}(x)\right)^2\left(\omega^l(x)\right)^2 dx \leqslant \|R_1^{2S}\|_{L^\infty}\int R_1^{2S}(x)\omega^{2l}(x)\,dx < \infty. \end{aligned}$$

Suppose K is a function on $\mathbf{R}^+$ satisfying $(K\cdot 0)$ and having $K(0) = 1$. We know then that for $k \in \underline{\mathscr{Q}}_l$ with $\hat{k} = K$, $k_t = \rho_t k$ is an approximate identity as $t \to 0$. If K satisfies $(K\cdot 4)$ then, using (1.2), Theorem 1.12 gives

$$|k_t(x)| \leqslant ct^{1/2}\left(t^{1/2}+|x|\right)^{-l} \qquad (0 < t \leqslant 1). \tag{1.17}$$

The following lemma is proved by a well-known technique. We include its proof here for completeness sake.

LEMMA 1.18. *For $s > 0$, let $B_s(x) = \{y \mid |y^{-1}x| < s\}$, and for $\phi \in L^1(G)$, let*

$$m^*\phi(x) = \sup_{s>0}|B_s(x)|^{-1}\int_{B_s(x)}|\phi(y)|\,dy.$$

If

$$f_t(x) = t^{-r/2}\left(1+t^{-1/2}|x|\right)^{-r-1},$$

then for some C and all $\phi \in L^1(G)$,

$$|f_t * \phi(x)| \leqslant cm^*\phi(x) \qquad (0 < t \leqslant 1).$$

PROOF. For some a, b and all $0 \leqslant t < 1$,

$$f_{t^2}(x) \leqslant \begin{cases} at^{-r} & \text{for } |x| < t, \\ bt\,|x|^{-r-1} & \text{for } |x| \geqslant t. \end{cases}$$

Thus, recalling that $B_s = B_s(e)$,

$$\begin{aligned} |f_{t^2} * \phi(x)| &\leqslant \int_{|y|<t} f_{t^2}(y)\,|\phi(y^{-1}x)|\,dy + \sum_{k=1}^{\infty} \int_{2^{k-1}t \leqslant |y| < 2^k t} f_{t^2}(y)\,|\phi(y^{-1}x)|\,dy \\ &\leqslant ac\,|B_t|^{-1} \int_{B_t} |\phi(y^{-1}x)|\,dy + \sum_{k=1}^{\infty} bt(2^{k-1}t)^{-r-1} \int_{B_{2^k t}} |\phi(y^{-1}x)|\,dy \\ &\leqslant ac\,|B_t|^{-1} \int_{B_t} |\phi(y^{-1}x)|\,dy + \sum_{k=1}^{\infty} 2^{-k+1} 2bc\,|B_{2^k t}| \int_{B_{2^k t}} |\phi(y^{-1}x)|\,dy \\ &\leqslant cm^*\phi(x). \end{aligned}$$

Since m^* is of weak type (1, 1) (cf. [**13**, Chapter III]), we have

COROLLARY 1.19. *If* $k \in \underline{\mathcal{C}}$ *such that* $\hat{k}$ *satisfies* ($K \cdot 4$), *then the maximal operator*

$$k^*f(x) = \sup_{0<t\leqslant 1} |k_t * f(x)|$$

is of weak type (1, 1).

We also have

COROLLARY 1.20. *If a function* K *satisifes* ($K \cdot 0$), *the operator*

$$T_{K_t} f = \int_0^{\infty} K(t\lambda)\,dE(\lambda)\,f$$

is bounded on all L^p, $1 \leqslant p < \infty$. *If*, *further*, $K(0) = 1$ *and* K *satisfies* ($K \cdot 4$) *with* $l = r + 1$,

$$\operatorname*{Lim}_{t \to 0} T_{K_t} f(x) = f(x) \quad (a.e., f \in L^p(G),\ 1 \leqslant p < \infty).$$

The following results concerning classical summation methods are obtained from Corollary 1.20. What could be called an Abel method is obtained by setting $K(\lambda) = e^{-\lambda}$. We get

COROLLARY 1.21.

$$\operatorname*{Lim}_{t \to 0} \int_0^{\infty} e^{\lambda t}\,dE(\lambda)\,f(x) = f(x) \quad (a.e., f \in L^p(G),\ 1 \leqslant p < \infty).$$

Noticing that for $N > (3r + 4)/2$,

$$K(\lambda) = \begin{cases} (1-\lambda)^N, & 0 \leqslant \lambda \leqslant 1, \\ 0, & \lambda > 1, \end{cases}$$

satisfies ($K \cdot 4$), we obtain a.e. convergence of the Bochner-Riesz means, which is

COROLLARY 1.22. *For* $N > (3r + 4)/2$,

$$\operatorname*{Lim}_{R \to \infty} \int_0^{R} (1 - \lambda R^{-1})^N\,dE(\lambda)\,f(x) = f(x) \quad (a.e., f \in L^p(G),\ 1 \leqslant p < \infty).$$

Finally, suppose that $\varphi \in C^N(\mathbf{R}^+)$ with $\varphi'(0) = 1$ such that

$$\sup_{\lambda \geqslant 0} |\varphi^{(j)}(\lambda)| < \infty \qquad (j = 0, 1, \ldots, N). \tag{1.23}$$

Setting $K(\lambda) = (\varphi(\lambda)/\lambda)^\alpha$ $(\alpha > 1 + (12 + 3r)S/2)$ and noting that K satisfies $(K \cdot 4)$, we obtain

COROLLARY 1.24.

$$\operatorname*{Lim}_{t \to 0} \int_0^\infty \frac{\varphi(t\lambda)}{\lambda} dE(\lambda) f(x) = f(x) \qquad (a.e., f \in L^p, 1 \leqslant p < \infty).$$

2. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. For $f \in L^1(G)$ set

$$\pi_f \alpha = \int_G f(x) \pi(x) \alpha \, dx$$

for each $\alpha \in \mathcal{H}$. Let $\mathcal{H}^\infty$ denote the subspace for all $\alpha \in \mathcal{H}$ such that $x \to \pi(x)\alpha$ is a smooth $\mathcal{H}$-valued function on G. There is a unique representation $X \to \pi_X$ of $\mathfrak{g}$ on $\mathcal{H}^\infty$ such that

$$\pi_X \alpha = (d/dt) \pi(\exp tX) \alpha_{|t=0}.$$

This representation extends uniquely to a representation of the universal enveloping algebra of $\mathfrak{g}$.

Suppose now that π is such that π_f is a compact operator for each $f \in L^1(G)$. Let $\beta_0, \beta_1, \ldots$ be an orthonormal basis for $\mathcal{H}$ such that $\pi_{p_1} \beta_n = a_n \beta_n$, where $a_n \in \mathbf{R}$. Since π_{p_1} is a positive operator with norm $\leqslant 1$, we write $a_n = e^{-\lambda_n}$. One then has that

$$\pi_{p_t} \beta_n = e^{-t\lambda_n} \beta_n = \hat{p}_t(\lambda_n) \beta_n,$$

and so, by definition of $\mathcal{Q}$, for all $f \in \mathcal{Q}$, $\pi_f \beta_n = \hat{f}(\lambda_n) \beta_n$. Since zero is the only possible accumulation point of $\{a_n\}$, $\{\lambda_n\}$ is a discrete (perhaps unbounded) sequence. We reorder the β_n's according to increasing values of the λ_n's. Clearly, the λ_n's are eigenvalues of π_L. It follows that if $f \in \underline{\mathcal{Q}}$ such that $\hat{f}$ has compact support, then π_f is finite dimensional.

Now let Γ be a discrete cocompact subgroup of G. We fix a symmetric, connected, compact fundamental domain D of G/Γ. We write $\dot{x} = x\Gamma$ and note that $x \to \dot{x}$ maps D one-to-one onto G/Γ. Let $\underline{\mathcal{H}} = L^2(G/\Gamma)$. If π is the quasiregular representation of G on $\underline{\mathcal{H}}$ and $f \in C_c(G)$, then π_f is an integral operator on $\underline{\mathcal{H}}$ with continuous kernel and, hence, compact. Thus π_f is compact for each $f \in L^1(G)$. Also, $\underline{\mathcal{H}}$ decomposes into a direct sum of irreducible subspaces $\underline{\mathcal{H}}_k$. Thus, we can select a basis $e_0, e_1, \ldots$ of $\underline{\mathcal{H}}$ with $e_j \in \underline{\mathcal{H}}_{k_j}$ such that

$$\pi_L e_j = L e_j = \lambda_j e_j \quad \text{and} \quad \lambda_j \leqslant \lambda_{j+1} \qquad (j = 0, 1, \ldots)$$

with e_0 being the constant one function and $\lambda_0 = 0$. For $f \in \underline{\mathcal{Q}}$ we then have

$$\pi_f e_j = f * e_j = \hat{f}(\lambda_j) e_j \qquad (j = 0, 1, \ldots).$$

Similarly, π_Δ has a discrete nonnegative spectrum $\xi_0 \leqslant \xi_1 \leqslant \cdots$ with corresponding eigenfunctions $h_0, h_1, \ldots$. Since π_Δ is elliptic, h_j is real analytic. (Note that for 2 step nilpotent groups, Δ and L commute, and hence $h_j = e_j$.) By virtue of (1.4) and

(1.4′) we have

$$((I+L)\varphi,\varphi) \leqslant ((I+\Delta)\varphi,\varphi) \leqslant C\big((I+L)^{\sigma}\varphi,\varphi\big),$$

and so

$$1+\lambda_j \leqslant 1+\xi_j \leqslant C(1+\lambda_j)^{\sigma}. \tag{2.1}$$

By a theorem of H. Weyl (cf. e.g. **[14]**) $\#\{\xi_j \mid \xi_j \leqslant N\} \sim CN^{d/2}$. Thus, by (2.1) we get

$$C_1 N^{d/2} \leqslant \#\{\lambda_j \mid \lambda_j \leqslant N\} \leqslant C_2 N^{d/2\sigma}.$$

Let $k \in \underline{\mathcal{Q}}$ and set $k_t = \rho_t k$. Then

$$\pi_{k_t}\varphi = \sum_j \hat{k}(t\lambda_j)(\varphi, e_j)e_j. \tag{2.2}$$

But one also has

$$\pi_{k_t}\varphi(\dot{x}) = k_t * \varphi(\dot{x}) = \sum_{\gamma\in\Gamma}\int_D k_t(x\gamma y)\varphi(y^{-1})\,dy. \tag{2.3}$$

THEOREM 2.4. *If K satisfies $(K\cdot 0)$ with $l=0$, then*

$$\lim_{t\to 0}\Big\|\sum_j K(t\lambda_j)(\varphi,e_j)e_j - \varphi\Big\|_{L^p} = 0 \qquad (\varphi\in L^p(G/\Gamma),\ 1\leqslant p<\infty),$$

and

$$\lim_{t\to 0}\Big\|\sum_j K(t\lambda_j)(\varphi,e_j)e_j - \varphi\Big\|_{C(G/\Gamma)} = 0 \qquad (\varphi\in C(G/\Gamma)).$$

To prove a.e. convergence of (2.2) we need the following lemmas.

LEMMA 2.5. *Let $\Gamma_N = \{\gamma\in\Gamma \mid |\gamma| \leqslant N\}$. Then $\sup_{N\geqslant 1}(\#\Gamma_N)N^{-r} < \infty$.*

PROOF. If $x=\gamma d$ where $\gamma\in\Gamma_N$, $d\in D$, then $|x|\leqslant C(|\gamma|+|d|)\leqslant C'N$ for $N\geqslant 1$. Thus $\#\Gamma_N|D| = |\Gamma_N D| \leqslant |B_{C'N}| = (C'N)^r$.

LEMMA 2.6. *If k satisfies $(K\cdot 4)$ with $l=r+2$, then there exist $M, c\in\mathbf{R}$ such that*

$$|k_t\varphi(x)| \leqslant 2ct^{1/2}\sum_{|\gamma|\leqslant M}\int_D \big(t^{1/2}+|x\gamma y|\big)^{-r-1}|\varphi(y^{-1})|\,dy$$

for $x\in D$ and $0<t\leqslant 1$.

PROOF. By (1.18)

$$|k_t(x\gamma y)| \leqslant ct^{1/2}\big(t^{1/2}+|x\gamma y|\big)^{-r-2}.$$

Let $\gamma_0\in\Gamma$ such that $|\gamma|\geqslant|\gamma_0|$ implies $\gamma\notin D\cdot D$. Then $\inf\{|x\gamma_0 y| \mid x, y\in D\} = a > 0$. Thus $(t^{1/2}+|x\gamma_0 y|)^{-r-2} \geqslant a^{-r-2}$. Let

$$\inf\big\{|x\gamma y|\,|\gamma|^{-1} x, y\in D, |\gamma|>|\gamma_0|\big\} = b > 0.$$

Then

$$\sum_{|\gamma|>|\gamma_0|} \left(t^{1/2} + |x\gamma y|\right)^{-r-2} < \sum_{|\gamma|>|\gamma_0|} (b|\gamma|)^{-r-2}.$$

By Lemma 2.5, there is an M such that $\Sigma_{|\gamma|>M}(b|\gamma|)^{-r-2} > a^{-r-1}$. Hence, assuming $M > |\gamma_0|$ we have

$$t^{1/2} \sum_{\gamma \in \Gamma} \left(t^{1/2} + |x\gamma y|\right)^{-r-1} \leqslant 2t^{1/2} \sum_{|\gamma| \leqslant M} \left(t^{1/2} + |x\gamma y|\right)^{-r-1}.$$

We identify $L^1(G/\Gamma)$ with the subspace of $L^1(G)$ consisting of those functions supported on D. Then

$$\text{(2.7)} \qquad k * \varphi(\dot{x}) = \sup_{0<t\leqslant 1} |k_t * \varphi(\dot{x})|$$
$$\leqslant 2c \sum_{|\gamma|\leqslant M} \sup_{0<t<1} \int_D t^{1/2}\left(t^{1/2} + |x\gamma y|\right)^{-r-1} |\varphi(y^{-1})|\, dy.$$

But

$$\int_D t^{1/2}\left(t^{1/2} + |x\gamma y|\right)^{-r-1} |\varphi(y^{-1})|\, dy = \int t^{1/2}\left(t^{1/2} + |xy^{-1}|\right)^{-r-1} |\varphi(y\gamma)|\, dy.$$

Hence, by Lemma 1.18,

$$\left|\left\{\dot{x} \,\middle|\, \sup_{0<t<1} \int_D t^{1/2}\left(t^{1/2} + |x\gamma y|\right)^{-r-1} |\varphi(y^{-1})|\, dy > \alpha\right\}\right| \leqslant (C/\alpha)\|\varphi\|_{L^1(D)}.$$

Thus, since the sum in (2.7) is finite, k^* is of weak type (1, 1). This gives

THEOREM 2.8. *If K satisfies $(K \cdot 4)$ with $l = r + 2$, then*

$$\lim_{t\to 0} \sum_j K(t\lambda_j)(\varphi, e_j)e_j(x) = \varphi(x) \quad (a.e., \varphi \in L^1(G/\Gamma)).$$

As before, we get analogues of the classical summation methods of Abel, Cesàro, and Riemann by considering specific functions K.

COROLLARY 2.9. *For $\varphi \in L^1(G/\Gamma)$,*

$$\lim_{\rho\to 1} \sum_j \rho^{\lambda_j}(\varphi, e_j)e_j(x) = \varphi(x) \quad (a.e.),$$

and

$$\lim_{n\to\infty} \sum_{j\leqslant n} \left(1 - \frac{\lambda_j}{n}\right)^N (\varphi, e_j)e_j(x) = \varphi(x) \quad \left(a.e., N > \frac{3r+6}{2}\right).$$

If ψ satisfies (1.23), and $\alpha > 1 + (14 + 3r)S/2$, we have

$$\lim_{t\to 0} \sum_j \left(\frac{\psi(t\lambda_j)}{t\lambda_j}\right)^{\alpha} (\varphi, e_j)e_j(x) = \varphi(x) \quad (a.e.).$$

3. Let G be the $(2k+1)$-dimensional Heisenberg group, i.e. $G = \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}$ with multiplication defined by $(\mathbf{x}, \mathbf{y}, z)(\mathbf{x}', \mathbf{y}', z') = (\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', z + z' + \mathbf{x} \cdot \mathbf{y}')$. Let X_j, Y_j be the elements of the Lie algebra of G corresponding to the one-parameter subgroups $t \to ((1, \dots, t, \dots), \mathbf{0}, 0)$ and $t \to (\mathbf{0}, (0, \dots, t, \dots), 0)$, respectively. Then $L = \Sigma_j X_j^2 + Y_j^2$. Let π^1 be the Schrödinger representation of G corresponding to the parameter $\lambda = 1$, i.e.,

$$\pi^1_{(\mathbf{x},\mathbf{y},z)} f(\mathbf{u}) = e^{i(\mathbf{y}\cdot\mathbf{u}+z)} f(\mathbf{x} + \mathbf{u}) \qquad \left(f \in L^2(\mathbf{R}^k)\right).$$

Let $\Gamma = \{(\mathbf{x}, \mathbf{y}, z) \mid \mathbf{x}, \mathbf{y} \in \mathbf{Z}^k, z \in \mathbf{Z}\}$. Then G/Γ is compact. The map $\tau\colon L^p_c(\mathbf{R}^k) \to L^p(G/\Gamma)$ $(1 \leqslant p < \infty)$, defined by

$$\tau f(\mathbf{x}, \mathbf{y}, z) = \sum_{\mathbf{m} \in \mathbf{Z}^k} e^{2\pi i z} e^{2\pi i \mathbf{m}\cdot\mathbf{y}} f(\mathbf{x} + \mathbf{m}),$$

extends to $L^p(\mathbf{R}^k)$ and is isometric on $L^2(\mathbf{R}^k)$ into $L^2(G/\Gamma)$. Also, $\tau\pi^1_{(\mathbf{x},\mathbf{y},z)} f = \pi_{(\mathbf{x},\mathbf{y},z)} \tau f$, where π is the quasiregular representation of G on G/Γ. One has that

$$\pi^1_L = \sum_j \left(\frac{\partial}{\partial u_j^2} - u_j^2 \right).$$

For $m = 0, 1, \dots$ let H_m be the mth Hermite function, and if $\mathbf{m} = (m_1, \dots, m_k)$, $j_m \geqslant 0$, $|\mathbf{m}| = \Sigma m_j$, $H_{\mathbf{m}}(\mathbf{x}) = H_{m_1}(x_1) \cdots H_{m_k}(x_k)$, then

$$\pi^1_L H_{\mathbf{m}} = (2|\mathbf{m}| + k) H_m.$$

Thus, setting $e_{\mathbf{m}} = \tau H_m$, if K satisfies $(K \cdot 4)$ with $l = r + 2$, we get, for $\varphi \in L^p(G/\Gamma)$, $1 \leqslant p < \infty$,

$$\lim_{t \to 0} \sum_{\mathbf{m}} K((2\mathbf{m} + k)t)(\varphi, e_{\mathbf{m}}) e_{\mathbf{m}}(\mathbf{x}, \mathbf{y}, z) = \varphi(\mathbf{x}, \mathbf{y}, z) \quad \text{(a.e.)}. \tag{3.1}$$

Putting $\varphi = \tau f$, and noting that $(\tau f, \tau H_m) = (f, H_m)$, we get

$$\sum_m e^{2\pi i \mathbf{m}\cdot\mathbf{y}} f(\mathbf{x} + \mathbf{m}) = \lim_{t \to 0} \sum_{\mathbf{m}} K((2|m| + k)t)(f, H_{\mathbf{m}}) \sum_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{y}} H_{\mathbf{m}}(\mathbf{x} + \mathbf{n})$$

for almost all $(\mathbf{x}, \mathbf{y}) \in [0,1]^k \times [0,1]^k$. Since (3.1) is also convergent in L^1-norm, multiplying by $e^{2\pi i \mathbf{j}\cdot\mathbf{y}}$ and integrating with respect to $\mathbf{y}$, we get, for almost all $\mathbf{x} \in [j_1, j_1 + 1] \times \cdots \times [j_k, j_k + 1]$,

$$f(\mathbf{x}) = \lim_{t \to 0} \sum_{\mathbf{m}} K((2|m| + k)t)(f, H_{\mathbf{m}}) H_{\mathbf{m}}(\mathbf{x}).$$

Putting specific functions for K, we get Abel, Cesàro and Riemann summability theorems for the Hermite expansion. The estimates on the exponent in the Riesz means do not seem to be the best possible and, as is known for norm convergence (which, of course, also follows), much smaller exponents are sufficient (cf. **[9]**).

4. Let G be an arbitrary nilpotent simply connected Lie group and let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$. For a positive-definite matrix (a_{ij}) we write $-\Delta = \Sigma a_{ij} X_i X_j$, and we note that Δ is invariant under any orthogonal change of the basis $X_1, \dots, X_n$. Let $X_1^*, \dots, X_n^*$ be the dual basis of $\mathfrak{g}^*$ and let $|\delta|^2 = \Sigma \delta_j^2$, if $\delta = \Sigma \delta_j X_j^*$. It is clear that the norm $|\cdot|$ depends only on the operator Δ.

According to Kirillov theory, the unitary irreducible representations π^δ of G are in a one-to-one correspondence with the coadjoint orbits $\mathrm{Ad}_G^* \delta$, $\delta \in \mathfrak{g}^*$.

The following theorem has been proved with great help from Roger Howe.

THEOREM 4.1. *For every δ in $\mathfrak{g}^*$ we have*

$$\min\{\lambda: \lambda \in \operatorname{Sp} \pi_\Delta^\delta\} \geqslant \min\{|\xi|^2: \xi \in \mathrm{Ad}_G^* \delta\}.$$

PROOF. We proceed by induction on $\dim G$. If G is Abelian, the theorem is trivial (and, in fact, equality holds). By usual reductions we may assume that the center of $\mathfrak{g}$ is one dimensional, say $\mathbf{R}Z$, and that $\langle \delta, Z\rangle \neq 0$. We select an element Y in the hypercenter of $\mathfrak{g}$ but not in the center, and we write $\mathfrak{h} = \{W: [W, Y] = 0\}$. Let X be orthogonal to $\mathfrak{h}$ and of length one (note that the basis $X_1, \ldots, X_n$ defines an inner product in $\mathfrak{g}$). We easily check that $\mathfrak{g} = \mathfrak{h} \oplus \mathbf{R}X$.

Let δ_1 be the restriction of δ to $\mathfrak{h}$ which we also regard as a functional on $\mathfrak{g}$ putting $\langle \delta_1, X\rangle = 0$. Let $X^* = \delta - \delta_1$. We note that

$$\mathrm{Ad}^*_{\exp tY}\delta = \delta_1 + (1 + t\langle \delta_1, Z\rangle)X^*,$$

whence

$$\mathrm{Ad}_G^*\,\delta = \mathrm{Ad}_G^*\,\delta_1 + \mathbf{R}X^*.$$

Consequently,

$$\min\{|\xi|^2: \xi \in \mathrm{Ad}_G^*\,\delta\} = \min\{|\eta|^2: \eta \in \mathrm{Ad}_G^*\,\delta_1\}. \tag{4.1}$$

We note that $\exp \mathfrak{h} = H$ is a normal subgroup of G and also that every maximal subalgebra of $\mathfrak{h}$ subordinate to δ_1 is a maximal subalgebra of $\mathfrak{g}$ subordinate to δ. Thus π^δ is the induced representation of π^{δ_1} of H. Since $G/H = \mathbf{R}$, we let

$$\mathcal{H}(\pi^\delta) = \left\{f: \mathbf{R} \to \mathcal{H}(\pi^{\delta_1}) \int \|f\|^2 < \infty\right\}.$$

We write $\Delta = \Delta_1 - X^2$, where $-\Delta_1 = \sum X_j'^2$ for an orthonormal basis $X_1', \ldots, X_{n-1}'$ of $\mathfrak{h}$.

We have

$$\begin{aligned}\min\{\lambda: \lambda \in \operatorname{Sp} \pi_\Delta^\delta\} &= \inf\{(\pi_\Delta^\delta f, f): \|f\| = 1\}\\ &= \inf\{(\pi_{\Delta_1}^\delta f, f) + (\pi_X^\delta f, \pi_X^\delta f): \|f\| = 1\}\\ &\geqslant \inf\{(\pi_{\Delta_1}^\delta f, f): \|f\| = 1\}.\end{aligned}$$

But

$$\begin{aligned}(\pi_{\Delta_1}^\delta f, f) &= \int (\pi_{\Delta_1}^\delta f(t), f(t))\,dt = \int (\pi_{\Delta_1}^{\delta'_1} f(t), f(t))\,dt\\ &\geqslant \inf\{(\pi_{\Delta_1}^{\delta'_1}\phi, \phi): t \in \mathbf{R}, \phi \in \mathcal{H}(\pi^{\delta_1}), \|\phi\| = 1\}\\ &\geqslant \inf\{\lambda \in \operatorname{Sp} \pi_{\Delta_1}^\eta, \eta \in \mathrm{Ad}_G^*\,\delta_1\},\end{aligned}$$

where $\delta_1^t = \mathrm{Ad}^*_{\exp tX}\delta_1$.

By inductive hypothesis and (4.1), we have

$$\begin{aligned}\inf\{\lambda: \lambda \in \operatorname{Sp} \pi_{\Delta_1}^\eta, \eta \in \mathrm{Ad}_G^*\,\delta_1\} &\geqslant \min\{|\eta|^2: \eta \in \mathrm{Ad}_G^*\,\delta_1\}\\ &\geqslant \min\{|\xi|^2; \xi \in \mathrm{Ad}_G^*\,\delta\},\end{aligned}$$

which completes the proof.

Let

$$C^l(G/\Gamma) = \{f: (1+\Delta)^{l/2} f \in L^2(G/\Gamma)\}.$$

We again write

$$L^2(G/\Gamma) = \bigoplus \sum \mathcal{H}_k, \tag{4.2}$$

where for each k the quasiregular representation of G is an irreducible representation π^{δ_k} on $\mathcal{H}_k$. Let P_k be the projection on $\mathcal{H}_k$.

The following corollary exhibits a link between smoothness of a function f in $L^2(G/\Gamma)$ and the size of $\|P_k f\|_{L^2(G/\Gamma)}$ in terms of the geometry of the dual of G which appears in the decomposition (4.2).

COROLLARY 4.2. *If $f \in C^l(G/\Gamma)$, then*

$$\|P_k f\|_{L^2(G/\Gamma)} \leq C_f\left[\min\{|\delta| : \delta \in \mathrm{Ad}_G^* \delta_k\}\right]^{-l},$$

where C_f is independent of k.

PROOF. If $\lambda_k = \min\{(\Delta f, f): f \in \mathcal{H}_k, \|f\| = 1\}$, then

$$\|\Delta^{l/2} f\|_{L^2(G/\Gamma)} \geq \|\Delta^{l/2} P_k f\|_{L^2(G/\Gamma)} \geq \lambda_k^{l/2} \|P_k f\|_{L^2(G/\Gamma)}.$$

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